IN SEARCH OF STRATEGIES USED BY PRIMARY SCHOOL PUPILS FOR DEVELOPING FRACTION SENSE

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ABSTRACT

Purpose – Most literature has focused solely on either knowledge about number sense or understanding of fractions. To fill the research gap, this study examined pupils' abilities in both number sense and fractions. In particular, it investigated Year 4 and Year 5 pupils' use of strategies in developing their fraction sense.

Methodology – This study adopted a descriptive research design, utilising a mixed approach in data collection. An instrument called the Fraction Sense Test (FST) and a clinical interview were used to collect data. The FST comprised 3 strands: fraction concept, fraction representation and effect of operation. A two-stage cluster sampling method was employed to select 396 Year 4 and Year 5 pupils from a district in Selangor, Malaysia. The sampling involved random selection of the primary schools in the first stage, followed by pupils within the selected schools in the second stage. In addition to descriptive statistics, content analysis of interview transcripts was conducted to identify the presence of concepts and strategies applied among the pupils.

Findings – The study found that the pupils scored lowest in effect of operation. It was also revealed that there were four strategies which helped the pupils to develop fraction sense, namely (1) comparing fractions using benchmark fractions of common fractions such as ½, ¼, zero and 1, (2) understanding denominators to determine the size of equal parts, (3) comparing fractions using unit fraction, and (4) applying the strategies in (1) and (2) to manipulate fractions in effect of operation.

Significance – The findings provide useful input to facilitate the development of fraction sense ability.

Keywords: Fraction sense, fraction concept, fraction representation, effect of operation.

INTRODUCTION

Numeracy is taught in the early stage of childhood education, usually through play and simulations, with the objective of achieving a certain numerical competency level gradually. When children are fluent and comfortable in using and manipulating numbers, they would demonstrate the ability to work with numbers in various situations, including problem-solving. This would provide them with the opportunity to revisit and venture deeper into various mathematical problems and uses in different contexts (Mukwambo, Ngcoza, & Ramasike, 2018; Rohrer, 2009). Students learn from examples to develop a conceptual understanding of mathematics, which would enable them to be flexible in thinking. As their development of mathematical flexibility in thinking progresses, students will develop what is known as number sense. It is therefore appropriate to introduce fractions during the development of number sense so that students will become comfortable and fluent in using and manipulating fractions (Yang, 2003). Fractional concept development begins as early as the time children explore shapes. Hart (1981) drew early attention to the idea that the acquisition of knowledge in fractions is difficult to be achieved by the majority of children. The importance of gaining fractional knowledge is further explained by Roesslein and Codding (2019) and Hughes (2019). It is highlighted that proficiency and a deep understanding of fractions are essential for secondary school students. Gaining fractional knowledge is especially advantageous because fractions are foundational to many more advanced mathematics knowledge

(Hoon, Yaakob & Singh, 2016; Hughes, 2019; Siegler, 2017; Soni & Okamoto, 2020). Therefore, it is essential to give appropriate attention to the development of skills and knowledge of fractions among students.

Fraction sense ability is developed when repetitive ability is demonstrated through the sharing of the understanding of fraction concepts, representation with comparison, and the operations of dealing with fractions (Mukwambo et al., 2018). Notably, fraction sense implies a deep and flexible understanding of fractions that results from the ability to reason about fractions, rather than applying rules and procedures blindly (Kilpatrick, Swafford, & Findell, 2001; McNamara & Shaughnessy, 2015). The nature of development in fractions involves the sense of recognising partition of shapes and equal shapes. Then, the meaning of fractions in terms of unit fractions (fractions with numerator 1) and comparing fractions build the basis of students' understanding of the fraction concepts (McCoy, Barnett, & Stine, 2016). A sense of equivalence is later developed, which contributes to the understanding of complexity in fractions. The understanding is much more than recognising the numerator and denominator in terms of parts of a given complete partition. It involves all the possible concepts in different representations of fractions (Clarke, Roche, & Mitchell, 2007; Lamon, 2012). Hence, to be competent in fractions, pupils need to discover fractional sense in terms of number concept and multiple representations for fractions equal to a certain standard unit. It may be that one unit plays a very important role in building up the knowledge (Namkung, Fuchs, & Koziol, 2018). In understanding the number concept, it is imperative that students have the understanding of fractional parts from the basis of fractions which shows the representation of number in terms of parts. Then it becomes complex when operations are involved.

Similar to numerical sense, fraction sense refers to the use of various strategies based on the individual's ability in conceptual understanding and generalising the understanding of fractions. Nurturing fractional sense is vital since making sense and reasoning are integral in developing conceptual understanding and intensifying mathematical knowledge. A study by Nik Azis (1991) on students' schemes of fractions has highlighted the significance of practicing some strategies when learning fractions. Basically, strategies used for demonstrating fractional knowledge would focus on recognising a fraction situation to establish knowledge of proper or improper fractions based on part to whole concepts, comparing two fractions

and transforming a fraction situation into an equivalent situation (representation). Notably, students still need to develop fractional sense for a deep and flexible understanding of fractions. Analyses of studies have indicated that understanding of fractions among primary school students would be the most problematic area in both teaching and learning of mathematics (Booth & Newton, 2012; Chinnappan, 2005; Tian & Siegler, 2017; Wu, 2001). Nevertheless, students need more help to improve their ability to make sense of situations pertaining to number concepts (Akkaya, 2015; İymen & Duatepe-Paksu, 2015; Orhun, 2007; O'Connor, 2001; Yetim & Alkan, 2010).

Most literature has focused solely on either number sense or understanding of fractions. Since both understanding of fractions and ability in number sense make a major contribution in the development of science, technology, mathematics and engineering (STEM), there is a need to further investigate how fraction sense (which is the integrated knowledge of fraction and number sense) is developed, particularly in terms of strategies used among the students (Becker & Park, 2011; Lamon, 2012; Philipp, 2000). Among the researchers in this area, Altay and Erhan (2017) and Sengul (2013) have emphasized the use of strategies for the development of knowledge in fractions as well as number sense. To fill the gap in previous research, this study takes into consideration pupils' ability to apply strategies in number sense, focusing on one major topic, which is fractions. The scope in numeracy development covers number sense which directly involves fraction sense, which entails monitoring students' understanding through providing guidelines to help them develop the targeted knowledge. Therefore, this study focused on the acquisition of fractional knowledge based on fractional sense ability in the dimensions of number concepts, multiple representation and effect of operation. Specifically, this study investigated Year 4 and Year 5 pupils' use of strategies in developing their fraction sense.

METHODOLOGY

Population and Sampling

The population of this study was all Year 4 and Year 5 primary pupils in a district of Selangor, Malaysia. There are 61 primary schools in this district with 10,607 pupils studying in Year 4 and Year 5 at the time of study. A two-stage cluster sampling was employed. In

common practice, a two-stage cluster sampling involves selecting samples from two stages, with random selection of naturally existing clusters in the first stage, and random selection of samples within clusters in the second stage (Fraenkel, Wallen, & Hyun, 2012). In this study, selection for the first stage was based on clusters, i.e., the schools. In the second stage, the pupils within the selected schools were randomly selected. The clusters were from five different schools out of the 61 schools. Within the five schools (cluster), a simple random sampling was implemented to select the targeted samples. They comprised a total of 396 Year 4 and Year 5 pupils. The two-stage cluster sampling was implemented since the schools were mutually homogeneous in terms of the system of management in conducting classes and processes of learning. Yet, within the school, internally heterogeneous groupings were observed among different classrooms. The rationale for using primary pupils is based on Kass and Maddux's (2005) developmental theory of learning, which suggests that students develop the ability to visualize abstract information and internalize concepts between the ages of 8 and 11. Conceptually, Year 4 and Year 5 pupils should be able to represent fraction sense problems by translating the linguistic information and developing schematic visuals of the problems. In addition, the participants were homogeneous in respect of educational background since they had been learning Mathematics for at least 4 years from preschool to primary level. Meanwhile, six pupils (respondents) were selected to participate in the interview. Two respondents each were chosen from the high, intermediate and low scorers of the Fraction Sense Test.

Instrumentation

A descriptive research design utilising mixed approach data collection was conducted. Both quantitative and qualitative approaches were used to obtain sufficient data to address the research objectives and questions. The instrumentation comprised two measurements in two different phases. To collect the quantitative data, an instrument called the Fraction Sense Test (FST) was adapted from the original "Number Sense Test" designed by McIntosh, Reys and Reys (1992). The test has also been adapted by Noor Azlan Ahmad Zanzali and Munirah Ghazali (1999) and Parmjit (2009) to test number sense abilities among students. The instrument used in this study consisted of 16 items focusing on strands of 'understanding and use of the meaning and size of fractions' (Fraction Concept), 'understanding and use of equivalent forms and representations of fractions'

(Fraction Representation), and 'understanding the meaning and effect of operations' (Effect of Operation) (Kor, Teoh, Siti Syardia Erdina, & Parmjit, 2019). To ensure the validity of the FST, an inter-rater member check was performed to determine the accuracy of the instrument. An inter-rater member check involves a process where a rater other than the researcher is asked to verify the problem in accordance with the category. For this research instrument, four experts who are Mathematics professors from three universities and an accomplished language teacher were involved in validating the appropriateness of the language. The stability of the content was established via test-retest reliability. A total of 15 primary school students took a pilot test comprising a pre-test and a post-test. A correlation between the pre-test and post-test was identified to test the reliability of the pilot testing. The findings indicated a high and significant correlation ($r = 0.914$, $p < .05$) from the test-retest reliability, implying an elevated level of reliability.
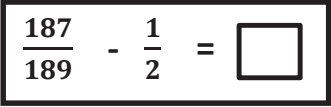
Data Collection

During quantitative data collection, the respondents were provided with an answer sheet. The questions, however, were projected on a white board to avoid paper-and-pencil computation by the respondents. The answer sheets were collected and evaluated. Each correct or incorrect item was scored with 1 or 0 point respectively. The questions on the FST did not require complex or long calculations but relied on the respondents' ability to make sense of the fractional situation, to apply their knowledge of fractions and operations and to use insight and/or estimation to arrive at the answers. The test items were divided according to the number sense strands.

In the qualitative data collection, the first round of interviews with the six respondents involved a semi-structured interview which consisted of the 16 questions in the FST. The purpose of this interview was to gain a detailed representation of a respondent's perceptions or accounts of a particular topic (De Vos, Strydom, Fouche, & Delport, 2002). The reason for asking all the questions in the FST was to identify the items that could provide the most information about the respondents' abilities and strategies in solving the FST questions. An interview schedule was prepared to serve as a guide to the overall issues covered in the interview, namely to determine the cognitive thinking in solving fraction problems and the strategies that the students used to answer fraction sense based questions. Think-aloud interviewing was also applied to prompt students to provide support

and explanations for their responses (Patton, 2002). This method encourages participants to verbalize their thinking and permits the researcher to pose any necessary leading questions throughout the interview. It is advantageous not only to the researcher in gathering more detailed data but also benefits the students in that the prompting promotes the development of scaffolding for the students' pursuit of knowledge in fractional number sense (Sarama, Clements, Swaminathan, McMillen, & Gonzalez Gomez, 2003). For the second round of interviews, the clinical interview was performed to get deeper into the respondents' cognitive understanding of fractions and their strategies in fraction sense. Table 1 illustrates the items were used in the clinical interview, based on the responses received from the first round of interviews.

Table 1

Selected Items for the Clinical Interview

Item No.	FST Strands	Questions
2	Fraction Concept	Which fraction is the closest to $\frac{1}{2}$? A. $\frac{1}{8}$ B. $\frac{4}{10}$ C. $\frac{1}{12}$ D. $\frac{1}{7}$
3	Multiple representation	Choose the fraction which best represents the amount of the box shaded. A. Slightly less than one-fifty B. Slightly less than half C. Slightly less than three-quarter D. More than three-quarter
5	Effect of Operation	$\dfrac{187}{189} - \dfrac{1}{2} = \square$ A. Less than $\frac{1}{2}$ C. Greater than $\frac{1}{2}$ B. Exactly 1 D. Impossible without calculating

Data Analysis

In the analysis of quantitative data, descriptive statistics was applied to obtain the numeric details, whereas in the analysis of qualitative data, content analysis was conducted to understand the information conveyed by the interviewees (Krippendorff, 2018). The latter approach allows any codes to emerge during the data analysis (Elliott, 2018; Tesch, 1990). In this study, the content analysis aimed to identify the presence of concepts and strategies applied among the respondents in fraction sense from the analysis of the texts in the transcripts.

RESULTS

The following analyses are discussed in relation to the research questions. Interpretations of the analyses highlight the findings in two main scopes, namely the students' achievement in fraction sense and their ways of understanding.

Research Question 1: What is the pupils' achievement in the FST according to the strands?

Results of the FST were analysed based on the number sense domains identified, namely: number concept/ fraction concept (Strand 1), multiple representations (Strand 2), effect of operations (Strand 3). Table 2 shows the descriptive statistics on the pupils' achievement in the FST according to the strands.

Table 2

Descriptive Statistics on Pupils' Performance in the FST According to Strands

	Strand 1 (fraction concept)	Strand 2 (multiple representations)	Strand 3 (effect of operations)
No. of question/total mark	7	4	5
Item	1,2,7,9, 11,14,15	3,6,10,12	4,5,8,13,16

	Strand 1 (fraction concept)	Strand 2 (multiple representations)	Strand 3 (effect of operations)
Total mark	7	4	5
Midpoint of the total mark	3.5	2	2.5
Mean (m)	2.35	1.63	1.32
Median	2.00	2.00	1.00
Standard deviation (s)	1.33	1.06	0.96
Percentage %	33.57	40.75	26.40
Sample size= 396			

Table 2 shows that the means for Strand 1, Strand 2 and Strand 3 were $m_{strand\ 1}$=2.35 (s =1.33), $m_{strand\ 2}$=1.63 (s=1.06) and $m_{strand\ 3}$= 1.32 (s=0.96) respectively. All these means are lower than the midpoints ('3.5' for Strand 1, '2.0 'for Strand 2 and '2.5' for Strand 3). In this study, pupils scored the lowest in strand 3 (effect of operation) in the FST.

Research Question 2: What is the pupils' achievement in the FST according to the test items?

This section details the results of the pupils' achievement in FST according to the test items by providing further description of the results exhibited in Table 2. The results in the first strand (Fraction Concept) showed that the performance for Items 1 and 11 were at a medium level, while Items 9 and 14 were slightly below the average level. Based on the performance, Items 2, 7 and 15 seemed to be tough for the pupils. For these items, the means recorded were below 0.3. It was noticed that the pupils had a certain difficulty level in understanding the concept of fractions. In the second strand (Multiple Representations), results for Items 6 and 10 were slightly below average. However, a level of difficulty was seen in Items 3 and 12. The means for both items were below 0.4. The results obtained from this strand revealed that pupils generally had a low level of ability to recognise fractions in multiple ways. For the third strand (Effect of Operation), the pupils exhibited a massive struggle for all the items concerned. Items 4, 5, 8 and 13 obtained slightly lower means, i.e., under 0.3. Meanwhile, Item 16 scored slightly higher (m

$=0.36$, s$= 0.479$), but was still under average. It was evident that the pupils had great difficulty in understanding this particular fraction strand.

Research Question 3: How do the pupils apply strategies in solving fractions?

Pupils' understanding of fraction sense was observed from the strategies used in solving the questions. Six respondents took part in the clinical interview.

Strand 1 (Fraction Concept)

The item in Figure 1 appeared as an item in Strand 1. This question aimed to explore respondents' sense of magnitude in fractions using $\frac{1}{2}$ as a referent.

Item 2

   Which fraction is the closest to $\frac{1}{2}$?

   Pecahan manakah yang paling hampir dengan $\frac{1}{2}$?

   A. $\frac{1}{8}$ C. $\frac{1}{12}$

   B. $\frac{4}{10}$ D. $\frac{1}{7}$

Figure 1. An item in Strand 1.

Referring to the quantitative analysis of the FST, the percentage of correct answers for Item 2 was only 15%, which was also the second lowest score obtained by the pupils among the 16 items. Results of the interview indicated that out of the six selected respondents, five answered this question incorrectly and only one was able to get the correct answer based on conceptual understanding which involved written calculation. After several series of probing, it was found that the five respondents who provided the incorrect answers had earlier encountered some problems in identifying the size of non-unit fractions. Among them, R1 and R2, for example, were found incapable of drawing even partitioned circles of fractions. Initially, all the respondents could comprehend the question, but they took a long time to discern the answer. They defended their answers with skepticism, and did not offer the fraction sense approach, as shown below.

I : What is your answer and how do you derive the answer?

R1 : My answer is A. I multiplied with…(silent) with… I'm not sure.

R3 : I think the answer is D because all the other fractions are bigger than one-half.

R4 : I don't know the answer. I assumed the answer is D.

Responses from the interviews confirmed that the respondents were unable to identify the closest fraction to one-half, suggesting that they did not understand the magnitude of fractions and were unable to apply their basic knowledge of equivalent fractions. However, one respondent was able to answer the question correctly, providing the answer with confidence, utilizing mental calculation.

From the attempted explanation shown in the extract of R6's response below, it is clear that the respondent had a vague understanding of fractions, in which estimation and mental calculation were applied with benchmarking of common fractions, namely a half, in solving this question. R6 had mentally simplified the fraction by dividing the numerator and the denominator with the same number. Although R6 did not mention about the equivalent fraction, it was obvious that the respondent was aware of the steps taken, and managed to apply prior knowledge of fractions to solve this question. R6 then estimated $\frac{4}{10}$ as the closest answer to $\frac{5}{10}$ among the other options. In this case, R6 had determined a half for fraction with denominator 10 as $\frac{5}{10}$. Then, $\frac{4}{10}$ was used, to be compared with the benchmark of a half, namely, $\frac{5}{10}$.

R6 : I think the answer is B. Because if I change $\frac{4}{10}$ to $\frac{5}{10}$, the answer will be $\frac{1}{2}$. So I think B is the closest answer.

I : How do you know $\frac{5}{10}$ is equal to $\frac{1}{2}$.?

R6 : Because 5 divided by 5 is equal to 1, and 10 divided by 5 is equal to 2.

I : Do you mean equivalent fraction?

R6 : Yes, teacher! I forgot the term.

The differences in the respondents' responses prompted the interviewer to investigate their understanding of fraction size. The interviewer asked them to compare fractions with the same numerator but with a different denominator, a different numerator but the same denominator, and a different numerator/denominator. Their responses were fairly similar and best summarized by the extract from R3's response below.

I : Can you compare $\frac{1}{7}$ and $\frac{1}{8}$, which one has a bigger size?

R3: $\frac{1}{8}$. Oh… no, no! has the bigger size.

I : Alright. How about $\frac{1}{7}$ and $\frac{4}{7}$?

R3: $\frac{4}{7}$

I : How about $\frac{4}{10}$ and $\frac{1}{10}$?

R3: $\frac{4}{10}$ is bigger, teacher.

The respondents were also asked to see how they compared two fractions with a different denominator. The respondents presented their understanding by observing the denominator and numerator as two quantities. Their responses were fairly similar and best summarized by the extract of R3's response below.

I : Very well. Now please compare which fraction has a bigger size, $\frac{6}{8}$ or $\frac{5}{6}$?

R3: Hmm... I think $\frac{6}{8}$.

I :   Can you justify your answer?
R3:   (long pause) I'm not sure, teacher. I feel like '6 and 8' are bigger than '5 and 6'.
I : Alright, then how about $\frac{5}{10}$ and $\frac{8}{20}$?
R3: I think $\frac{8}{20}$.

The extracts from the interview revealed that fractional sense on the values of fraction was determined based on the denominator if the numerator was '1'. Students were able to have a sense of the values

by comparing either only by the denominator or the numerator. In addition, the interview highlighted that the pupils also had difficulty in comparing the values based on the difference of two fractions. Particularly, it was also detected that a fraction-sensible strategy involving equivalent fractions was also absent when students tackled the problems assigned to them. Knowledge is vital and would be very helpful in comparing the fractions such as having the thought that the half of 10 is 5, means 5/10 is the half. the half of 8 is 4 means 4/8 gives the half, and the closest to the half is always referred to the half of the denominator.

Further probing and checking were conducted on the respondents' knowledge of the size of fractions by making them put in order the fractions on a number line. Only R6 managed to put together the fractions correctly. R6 was also the only person who was able to provide the correct answer on a prior question. Nevertheless, all of the respondents encountered difficulties in solving the fractions problems provided to them. Figure 2 exhibits the incorrect order of fractions on a number line produced by R2.
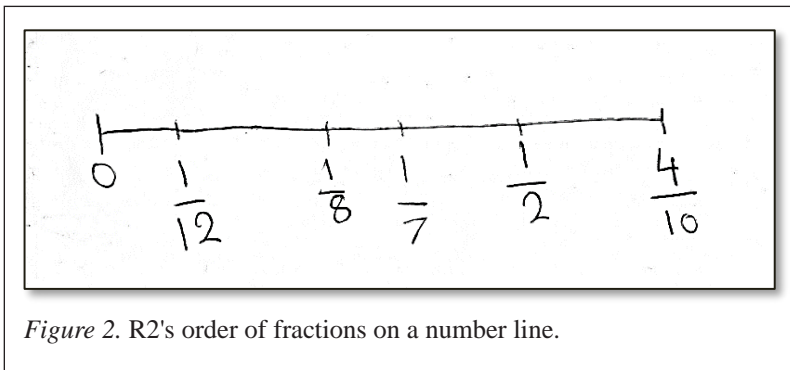

Figure 2. R2's order of fractions on a number line.

Figure 2 demonstrates that the respondent was able to order the unit fractions correctly. Nevertheless, the respondent was unable to work out the comparison between denominator and numerator correctly to determine the half. R2 indicated that $\frac{4}{10}$ is bigger than $\frac{1}{2}$.

When asked about the two values ($\frac{4}{10}$ and $\frac{1}{2}$), R2 indicated that the values '4' and '10' were bigger than '1' and '2'. Further investigation, as described in the following extract, revealed that the respondent understood the concept of unit fraction, but was unable to use

the same understanding with non-unit fractions. Subsequently, R2 failed to see the relationship between the denominator and numerator. Instead, the denominator and numerator were identified as independent quantities.

I : Are you sure this is the correct order?

R2: Erm… I'm not sure where to put $\frac{4}{10}$.

I : Check out this fraction again. Is there anything you can do to make this fractional number smaller?

R2: (Long pause). Divide the 'upper' and 'lower' number by 2.

I : So the answer is?

R2: $\frac{2}{5}$.

I : [So do you know where you should put $\frac{2}{5}$?

R2: (Shook her head).

The extract shows that R2 had acquired the knowledge of simplifying fractions by giving the correct equivalent fraction. However, when the output did not meet her very basic fraction knowledge, i.e., the unit fraction, R2 was perplexed and unable to order $\frac{2}{5}$ alongside other fractions on a number line. Regardless, the interviewer continued with further probing in order to identify if it was exceedingly difficult for this respondent to differentiate the relative size of fractions.

I : Look at these fractions. Which one is bigger, $\frac{5}{12}$ or $\frac{4}{10}$?

R2: (Silent). I can't imagine.

I : Now, let's compare $\frac{1}{12}$ with $\frac{1}{10}$ where the numerator is '1'. Which one is bigger?

R2: $\frac{1}{10}$ Because 1 is far different from 12 compared to 10

(referring to the numerator and denominator).

R2 later added:

R2: If I compare a cake which is divided into 10 parts with a cake divided into 12 parts, surely the cake which is divided into ten parts is bigger.

I : Then how about and $\frac{1}{2}$ and $\frac{4}{10}$?

R2: (Silence).

The interviewer encouraged R2 to present the fractions in graphical form after using the analogy of the cake portions. It aimed to evaluate how the respondent would partition a circle into an even number of parts. R2 found this task rather easy. R2 drew two circles, with one which was divided into two parts and another one into tenths. Subsequently, she shaded the parts according to what was asked in the question, as shown in Figure 3.
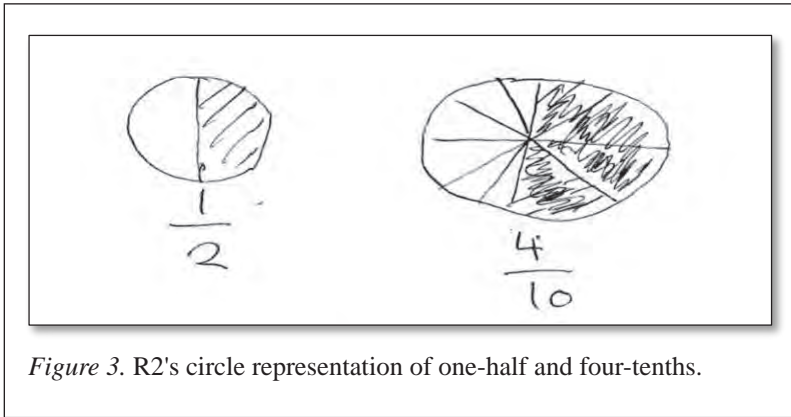

Figure 3. R2's circle representation of one-half and four-tenths.

The respondent then confirmed that $\frac{4}{10}$ was larger than $\frac{1}{2}$ as the shaded area of four-tenths was bigger than the other shaded area. Nevertheless, R2's drawing did not illustrate a precise fractional graphical representation. The uneven partitions of tenths led to the incorrect answer because accurate partitioning plays a critical role in fraction understanding (Behr, Lesh, Post, & Silver, 1983).

This finding was also similar to the experience of R1--despite managing to give the correct answer based on graphical comparison, the partitioning was inaccurate which resulted in failure to support the correct answer. The interviewer also asked R1 to show the drawing of $\frac{5}{12}$ just to re-evaluate the respondent's equal partitioning ability. Like R2, R1 also demonstrated the inability to draw an accurate or almost accurate partitioning of fractions. According to Wong and Evans (2011), it is crucial to start with an equally divided referent unit as errors can arise in naming the correct fraction since the significance of equal-sized parts is not recognised. Particularly, incorrect responses would disclose that pupils lacked understanding on the importance of a standard referent unit when comparing fraction quantities. The interviewer subsequently provided the respondents with equal partitioned circles with the insight gained from the previous sub-task, and asked them to shade the parts into

one-half and four-tenths. Unsurprisingly, all of them were able to shade the parts correctly according to the instruction.

Analysis of the interviews revealed that the pupils of low and intermediate ability levels memorised fractions by formula but not by the nature of fractions itself, where every part should be partitioned equally. The representations of fractions in graphical forms lacked uniformity in size. Pupils need to be provided with an accurate partitioned picture in order to perform correct fractions shadings or it may lead to incorrect answers due to poor portioning ability. This interview also provided evidence that the students were competent with unit fractions but were unable to compare the size of non-unit fractions. They could also determine if fractions were equal by giving the equivalent fraction or simplifying a fraction; however, they resisted using knowledge about equivalence to compare the size of fractions. Inadequacy of basic concepts in fractions deprived the pupils of being pertinent in fraction sense. Importantly, conceptual understanding of the values of fractions by comparing numerator and denominator requires representations or any guidance for the development of estimation and mental calculation. Therefore, pupils need to be exposed to various fractions situations to provide them with more experience in solving related mathematical problems. In this way, their fractional concepts acquisition may be developed from a few fraction situations.

Strand 2 (Multiple Representations)

The question for Strand 2 as shown in Figure 4 investigated whether the respondents could translate the diagrammatic representation of a fraction.

Item 10

Choose the fraction which best represents the amount of the box shaded.

A. Slightly less than one-fifth
B. Slightly less than half
C. Slightly less than three-quarter
D. More than three-quarter

Figure 4. An item in Strand 2.

Based on the quantitative analysis, only 38% of the pupils managed to provide the correct answer for this particular item. This was very much in agreement with the findings of the interviews in which only two of the six respondents, who were high achievers, were able to provide the accurate answer. It is interesting to note that pupils who were of the intermediate level were also able to provide the correct answers with good answering strategies using pencil and paper for calculation. However, the low achievers did not show any fraction-sensible strategies when answering this question and were unable to transliterate the multiple representations of fraction even if they were allowed to calculate using pencil and paper. The two high achievers who could provide the correct answers, i.e., R5 and R6, demonstrated interesting fraction sense strategies to solve this question. The following response was given by R6 during the interview session.

I : How did you get B as your answer?
R6: Err…I just imagined.
I : What did you imagine?
R6: I think if the shaded areas are placed next to each other, it will be easier to see how big the overall shaded area is. So, I drag this box (pointing at the bigger shaded area) to here (pointing at the gap between the two shaded areas)
I : So what is your answer?
R6: I think the answer is slightly less than a half.

R5, who also provided the correct answer, proposed quite a similar strategy, which was imagining the size of the total shaded area.

I : What is your answer?
R5: My answer is B.
I : How do you get that?
R5: I tried to divide the bar into six parts. From there I imagine the size of the shaded region. I think the region only takes about half of the whole diagram. So, I'm quite sure the answer is B.
I : Why did you choose six parts? Why not five or three?
R5: (Silence).
I : Never mind. Can you draw for me so that I can understand more?

R5 was then asked to sketch the 'imagined' fraction on paper and requested to draw on the question paper itself to have the exact image. The 'imagined' fraction is displayed in Figure 5.

Figure 5. R5's answer for Q5.
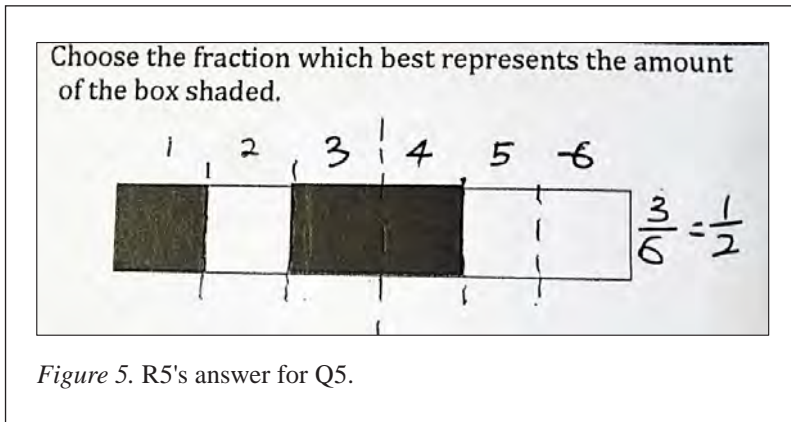
Based on Figure 5, it is obvious that R5 had clear steps to determine the size of the fraction. Firstly, the middle point of the bar was identified and marked as the midpoint. Based on the midpoint, the bar was divided into six equally partitioned parts which were then labelled at every part and the number of boxes/parts to be shaded was then determined. R5 proposed that "three boxes out of six have been shaded which is also equal to half of the whole diagram." Interestingly, the answer that R5 managed to get through the strategy used was not listed in the options provided in the item. Therefore,

the option closest to the answer managed in the calculation was selected.

The responses gathered from R5 and R6 displayed their abilities in fraction sense. Nevertheless, in order to determine whether these respondents really had the fraction-sensible strategies in multiple representations, further probing which was labelled as S2Q2 was carried out. The respondents were given a set of circles which were already shaded, and they were required to estimate the most suitable fraction for the circles. Figure 6 shows the images of the circles.

Figure 6. Further probing question (S2Q2).
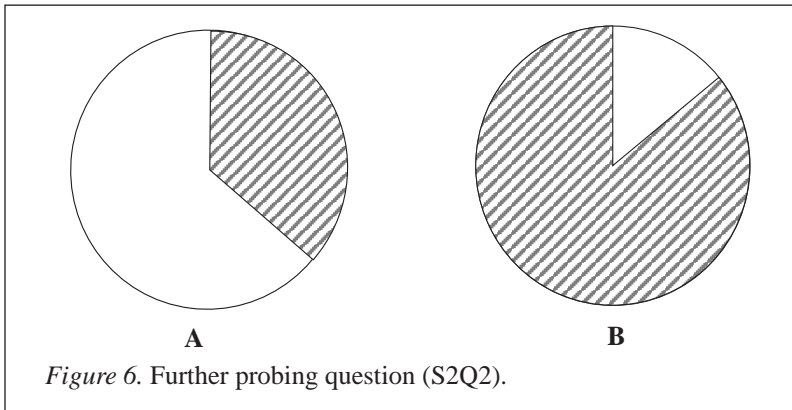
I : Can you estimate the fraction for the shaded region in A and B?

R6: Ermm... I think A is

I : How do you get that?

R6: I imagine that there's another line here (point to the image) which is the same size as the shaded area. When there's a line, it looks like this circle can be divided equally into 3 parts.

I : Very well, then how about B?

R6: (Pause for a while) Err… I think it's

I : How do you get ?

R6: I imagine this circle is divided into 8 parts. So the shaded area is only 7 parts.

From the extract, R6's responses show that the respondent could visualize fraction in multiple representations which also proves her abilities in fraction sense. However, a different situation was faced by the low achievers (R1 and R2) and the intermediate achievers (R3 and R4). All of them expressed a lack of confidence in answering this question, admitting that they did not know how to do it and just offered a wild guess. Their responses can best be illustrated by R2's statement below.

> I   :   What is your answer and how did you get the answer?
> R2:   My answer is A.
> I   :   Why did you choose A?
> R2:   Oh, no. I meant B.
> I   :   Okay, B. What's your reason for this?
> R2:   Because…Errm… I don't know…
> I   :   Did you just guess the answer?
> R2:   (Nodded).

Further probing (labelled S2Q3) was carried out, which required the respondents to represent $\frac{4}{5}$ in graphical form. Figure 7 shows the representation of $\frac{4}{5}$ produced by R2 in the form of a circle and a bar.
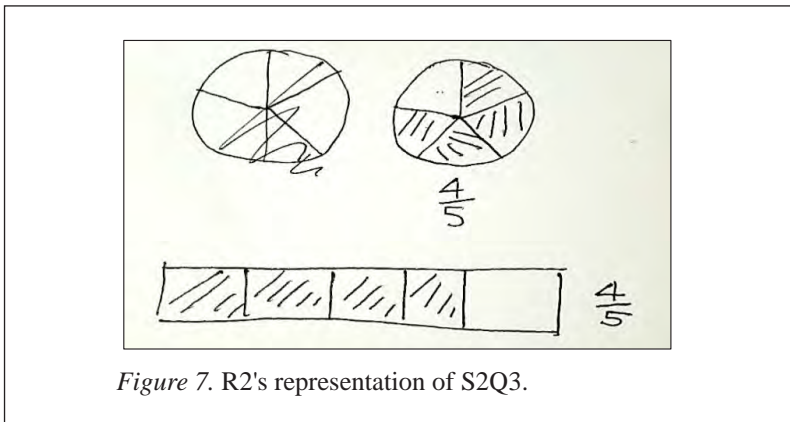

Figure 7. R2's representation of S2Q3.

Figure 7 indicates that R2 was able to represent the fractions in different diagrams and simultaneously propose the correct answer through the illustrations. The respondent was meticulous in drawing the diagrams, making sure that the parts were equal in proportion after realizing that an error was made in an earlier attempt, as seen in Strand 1. However, when a similar question was given, R2 was

unable to provide any estimation for the closest fraction as an answer.

Another representation of fraction (labelled S2Q4) is illustrated in Figure 8. This number line was adapted from the original McIntosh et al. instrument (question number 49) in the Number Sense Test. The purpose of this question was to analyse the respondents' ability to estimate or identify the size of fractions.
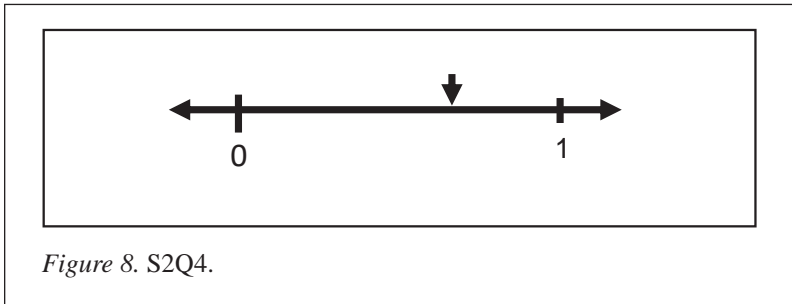

Figure 8. S2Q4.

As shown below, the responses indicated that R2's knowledge was limited to the usual representation of fraction commonly seen or learned in the textbook.

I   :   Can you estimate the value of fraction pointed by the arrow?

R2:   Err… I'm not sure. Less than one?

I   :   I'm sure you are getting closer. Can you be more specific?

R2:   (Long pause). I don't know…

I   :   Are you familiar with the number line? Have you seen it before?

R2:   Yes… I used to answer this kind of question but usually there are marks between the numbers so I can estimate the value.

R2 was unable to estimate the value pointed by the arrow because no other marks were available between the numbers to make an estimation of the value. Surprisingly, R2 understood that the value should be less than one but was unable to connect prior knowledge of simple unit fractions, such as 'a half' or 'a quarter', which would be helpful in estimating the value. Nevertheless, the

researcher encouraged R2 to form 'marks' by placing the values of $\frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ on the number line to help make a connection how these simple fractions could be of assistance in making the estimations. The work performed by R2 is illustrated in Figure 9.

Estimate the fraction shown by the arrow on the number line:

Figure 9. R2's answer for S2Q4.
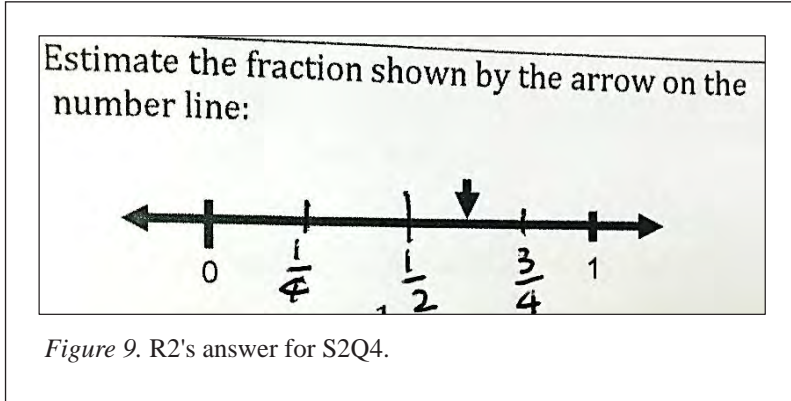
I : Now, based on the fraction marks that you've written, can you estimate the value of fraction pointed by the arrow?

R2: I guess it is less than $\frac{3}{4}$.

I : Why couldn't you guess that earlier?

R2: I could not imagine the marks. So I did not know how to estimate.

I : Are you more comfortable using paper and pencil to estimate the answer?

R2: Yes.

I : Can you propose a fraction that is suitable to be located in between $\frac{1}{2}$ and $\frac{3}{4}$?

R2: Err… I am not sure, teacher. It's difficult.

Evidently, the respondents were able to differentiate the size of the fractions because the fractions were common and would typically appear in their reference books. However, R1 and R2 who were categorized as low achievers were likely to be reliant on the use of the pencil and paper method to solve problems, which restricted their ability in fraction sense. Notably, responses provided by R2 in reference to the questions displayed in Figure 7 and Figure 9

revealed the respondent's obvious inability to deal with multiple representations of fractions and the privation of fraction sense skills.

On a more critical note, it should be pointed out that R2 was still unable to propose any suitable fraction to be located between $\frac{1}{2}$ and $\frac{3}{4}$. A similar situation occurred, as discussed in Strand 1, where the respondent was also unable to compare the size of different fractions, especially the non-unit fractions. However, unlike the low achievers, the intermediate achievers (R3 and R4) who were unable to guess the answer in the first attempt, were able to propose an acceptable fraction as the answer after managing some written calculations. Remarkably, R4 managed to provide an interesting solution, as shown in the following extract of the interview session.

I   :   Based on your calculation, what is the appropriate fraction to be put in between $\frac{1}{2}$ and $\frac{3}{4}$?

R4:   My answer is $\frac{7}{12}$.

I   :   How did you get the answer?

R4:   First, I equalised the denominators, so I multiplied $\frac{1}{2}$ with 2. Then I got $\frac{2}{4}$ and $\frac{3}{4}$. I still could not find the middle fraction. After that, I switched by multiplying both fractions by 3. I got $\frac{6}{12}$ and $\frac{9}{12}$. So, I think the fraction in the middle is $\frac{7}{12}$ or $\frac{8}{12}$.

Clearly, the respondent correlated prior knowledge of equivalent fractions to calculate the answer, having a clear perspective on the steps to solve the question. The respondent managed to arrive at the correct answer, but unfortunately was unable to solve it using fraction sense. Based on the interview, the high achievers demonstrated their ability to apply fraction-sensible strategies such as 'estimation' and 'imagination' when dealing with fraction problems. Their fraction sense aided them in visualizing fractions in multiple representations. While the intermediate achievers were unable to apply any fraction sense strategies, they demonstrated competency in solving fraction problems when they were allowed to use paper and pencil for their calculations. Of more concern was the low achievers' inability to display any fraction-sensible strategies and their persistent problem in identifying the size of fractions, even when they were allowed to

make calculations using pencil and paper. Such a condition may be caused by a lack of exposure to fraction concepts in Strand 1.

Strand 3 (Effect of Operation)

Item 5, as presented in Figure 10, investigates the respondents' ability to understand the effects of subtraction involving two fractions and the size of fractions.

Item 5 $\quad \dfrac{187}{189} - \dfrac{1}{2} = \boxed{}$

Figure 10. An item in Strand 3.

Based on the quantitative analysis, only 23% of the pupils managed to provide the correct answer to the question presented in Figure 10. This result was in accordance with the responses gathered in the interview, where only R5 was able to obtain the accurate answer using fraction sense, while the rest of the respondents found it difficult to make any computation without using pencil and paper. Further probing required the respondents to demonstrate their understanding through some written work. Although it was a matter of concern that R1 and R2 were unable to answer the question correctly, it was also interesting to discover that R3 and R4 were able to solve the problem but were unable to visualize the actual size of the fraction. However, R6 despite being unable to provide the answer to the problem in the first instance, provided a good fraction sense explanation for a similar question with smaller fractions at the end of the interview. On a brighter note, R5 was the first among the respondents to exhibit 'maturity' in identifying the answer using logic sense, as shown in the following extract of the interview session.

I   :   What is your answer?
R5:   My answer is A.
I   :   Why did you answer A?
R5:   Because…hmm… all of the other options do not make sense. If B, there is no way I can get exactly 1, as this is a subtraction question. The answer must be less. The

answer for C also seems illogical. If a half has been subtracted, there is no way the answer could be more than a half.

To affirm R5's ability in fraction sense, the interviewer then asked a question similar to Q5.

I : Without involving written calculation, can you estimate the answer for $\frac{88}{89} + \frac{3}{4}$?

R5: Hmm… I think it is around $1\frac{3}{4}$. Maybe less than that.

I : How did you guess that?

R5: My teacher always says when the numerator is the same as the denominator, the value is '1'. $\frac{88}{89}$ is almost 1. So I guess it must be less than $1\frac{3}{4}$.

R5 revealed not only an intuition on fractions but also the ability to comprehend the operation. R5 also managed to provide some rationalization for every option before providing an answer. Anghileri (2000) suggested that in order for children to develop good number/fraction sense, it is essential that they have a range of specific skills which includes estimation, good prediction and mental computation. In tandem with this, R5 demonstrated the ability to recall and acclimate prior knowledge about fractions in the appropriate situation. Nevertheless, the rest of the respondents stated that the question was challenging and impossible to be computed or even to guess the answer, as stated below.

R1: This is difficult, teacher. The fraction is too big. I can't answer it without calculating.

R2: Hmm… I can't solve this. I think I need to make some calculation.

R4: I'm not sure. I think I have to calculate.

I : Why do you need to calculate?

R4: Because… the fraction is too big.

The responses from R1, R2 and R4 revealed their belief that written calculation was the only method that could be used to solve the question. Further probing required these respondents to demonstrate their calculation and observe how their calculation would assist them in getting the final answer. R1 who was in the low performance

category responded that $\frac{182}{189} - \frac{1}{2} = \frac{186}{187}$. From the calculation made by R1, it was clear that the fraction was treated as two different numbers. R1 failed to make the connection between outcomes in a binary operation and the values carried in a fraction. It is important to note that R1 considered the fraction to be "too big" and that it was impossible to get the value for the denominator. Therefore, to gauge R1's ability to solve problems with fractions, some fraction problems were given with smaller figures. R1 was asked to solve $\frac{4}{5} - \frac{1}{2}$ and $\frac{3}{6} - \frac{1}{3}$. He responded that $\frac{4}{5} - \frac{1}{2} = \frac{8-5}{10} = \frac{3}{10}$ and $\frac{3}{6} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$.

I : Well done! Can you explain to me why you can answer these two questions but not the prior question?
R1: I don't know how to get the denominator.
I : Don't you think you can use the same method?
R1: No, because the fraction is a large figure.

The explanation given indicated that R1was able to solve simple fraction problems but had difficulty using the same method when attempting fractions with much larger figures. A similar question was directed to R4, who responded that $\frac{187 \times 2}{189 \times 2} - \frac{1}{2} = \frac{374-1}{378} = \frac{373}{378}$.

I : So, what is your final answer?
R4: $\frac{373}{378}$
I : Does this bring any meaning to you? Can you guess the answer?
R4: (Long pause). I cannot find the answer. I think the answer is unavailable.

However, the interviewer noticed the error made in R4's calculation and advised that the computation should be reassessed.

I : Are you certain about your calculation?
R4: Errr… I guess it's correct.
I : Can you please look again at your calculation? Take your time.
R4: Oops…I made a mistake! Can I amend?

R4 then amended the calculation and eventually provided the correct answer, which was $\frac{185}{378}$.

I   :   Now, does this help you in determining the answer for question 5?

R4:   (Long pause). I think it's C (greater than half).

I   :   Can you rationalize your answer?

R4:   Because this fraction is big, so it has to be greater.

From the above extracts of R1 and R4 responses, two hypotheses can be proposed: 1) the respondent is able to do written calculations involving fractions but tends to be careless, and 2) a big number in a fraction seems to distract respondents and restrict them from envisioning the actual size of the fraction. Further probing involving simpler fractions was carried out in order to corroborate these hypotheses. Except for R6, all the respondents proposed the same calculations and their responses can be best represented by the following extract from R3.

I   :   Can you solve $\frac{5}{6} - \frac{1}{3}$?

R3:   Can I calculate?
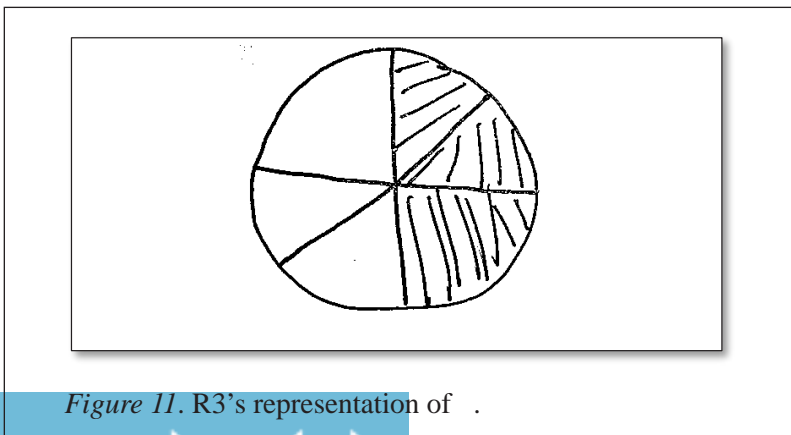
I   :   Yes.

R3:   The answer is $\frac{3}{6}$.

I   :   Are you confident with your answer?

R3:   Yes.

I   :   Can you draw a picture representing this fraction?

Figure 11 shows the graphical representation of $\frac{3}{6}$ produced by R3.

Figure 11. R3's representation of   .

I :   What can you see from this picture? I mean, how big is the area of the circle you have shaded?

R3:   Half of it.

I :   Does it mean 3/6 is equal to one-half?

R3:   (Paused for a while). Yes teacher! It is the same]

From the interview and the details provided in Figure 8, it is proven that the respondents were able to provide written calculations involving fractions by applying standard algorithm as taught in the classroom. Nevertheless, there might have been an inclination to make unintentional errors when they did not simplify their final answers. Continuous guidance and reminders from the teacher seemed to be indispensable for students to be able to obtain correct answers, despite having learned the necessary basic knowledge on the topic. However, continuous guidance and reminders are ineffectual because such actions do not improve pupils' abilities to envision or explain the actual size of fractions using fraction sense, and many are dependent on written calculations. Hence, the first hypothesis can be accepted. It should be noted though that only R6 was able to provide the correct answer to the question to prove the first hypothesis.

In order to prove the second hypothesis, the interviewer encouraged the respondents to estimate the answer for $\frac{9}{10} - \frac{1}{2}$ without using any written calculation. The interviewer emphasized on the word 'estimate' because the objective was to attain the closest figure to the correct answer, without forcing respondents to overthink with their calculation. Interestingly, four of the respondents, R3, R4, R5, and R6 were able to guess the answer accurately. However, R1 and R2 still found it impossible to guess the answer without making any written calculations. The following interview extract illustrates R6's rationalization in getting the answer to the problem.

R6:   I think, the answer must be less than a half.

I :   Why do you think so?

R6:   Because… $\frac{9}{10}$ is close to $\frac{10}{10}$ is 'one'. A half subtracted by one equals to half. But $\frac{9}{10}$ is less than 'one'. Therefore, the answer must be less than a half.

I :   Very well. Now, are you aware that this question is quite similar to question 5 I asked earlier? Why couldn't you answer that?

R6:   (Pauses for a while). The fraction is too big. It confuses me.

The explanation provided by R6 reveals that the respondent was able to utilise fraction-sense easily when it involved small fraction figures. The researcher then tested the respondent using a fraction with larger figures. Impressively, R6 was able to provide the accurate answer to the problem using logical fraction sense based on understanding the prior problem, which was simpler. R6's justification of the answer is provided in the following extract.

I : Now let's try with something similar to the original question. But this time we're going to use addition instead of subtraction.

R6: Alright.

I : Without involving any calculation, can you guess the answer for $\frac{187}{189} + \frac{1}{2}$?

R6: Errm… (Long pause). I guess the answer is less than $1\frac{1}{2}$.

I : Why do you say so?

R6: It is the same as just now. $\frac{187}{189}$ is close to $\frac{189}{189}$ which is one. 1 plus $\frac{1}{2}$ equals to $1\frac{1}{2}$. In this case, it must be less than that.

It is worth highlighting the respondents' persistent perception that it was impossible for them to solve fractional problems involving large figures without using any written calculations, which supports the second hypothesis, i.e., that large fractional figures are considered to be an obstacle, impeding their abilities to visualise the actual size of the fraction. Significantly, despite being allowed to write their calculations, the low performing students were persistent in their inability to attain the correct answers. This was caused by their lack of prior knowledge in fraction concepts. Also, the constant demonstration by the intermediate respondents that they were able to provide correct answers through written calculations but were unable to provide a clear explanation of their method, proves that they lacked fraction-sense in imagining the fractions. However, it was intriguing that the two high performing respondents demonstrated two varying abilities. R5 thought differently and managed to apply some fractional sense to obtain the correct answer, while R6, who was unable to guess the answer for the fraction with larger figures, could do so with small fractional figures.

DISCUSSION AND CONCLUSION

The results showed that pupils performed badly in the domain of "Effect of Operation" (Strand 3), indicating a lack of conceptual understanding of fractions and reinforcing the impression that their knowledge was limited to surface aspects and did not constitute a web of interconnected knowledge (Ma, 1999). This observation is in agreement with Nunes and Bryant's (2009) claim that most fractional representations have not been detected and well-studied. The pupils' lack of fractional sense could have been influenced by the fact that fractions do not form a 'normal' part of learners' day-to-day activities, except perhaps only the most common ones (Stewart, 2005), i.e., halves, quarters and thirds. Therefore, the pupils may not have had the necessary exposure to fractions with much larger figures. It is also believed that if the same practice of fraction sense in the estimation was integrated for similar problems, fraction sense ability could be developed. Henceforth, development of fraction sense would require the ability to recall or an awareness of mathematics patterns (in this case, fraction which is close to '1'), and the ability to relate a concrete symbol to a concept (in this case, the concept is half subtracted by one equals to half). The awareness of basic mathematics patterns is also closely related to fraction sense. Most of the respondents showed an understanding of standard written algorithm in fractions; however, the lack of fraction sense activities hindered them from being more versatile in adapting their prior knowledge of fractions, thus leading to the inability to come up with various flexible fraction-sensible strategies.

On the other hand, students with number sense develop useful and efficient strategies for managing numerical situations (Barrera-Mora & Reyes-Rodriguez, 2019; Reys et al., 1999; Reys & Yang, 1998); they are proficient mental calculators (Griffin, 2003) and good estimators (Sowder & Wheeler, 1989). Both proficient mental calculators and good estimators display flexible and adaptive use of strategies (Heinze, Star, & Verschaffel, 2009; Torbeyns, Verschaffel & Ghesquière, 2006). Hence, the first step towards development of fraction sense would be to focus on understanding the concepts without depending on calculators or paper and pencil. However, school mathematics teaching and learning still includes a considerable amount of work with paper and pencil skills. The pupils' performance on the test using paper and pencil was not reflecting their abilities to apply flexible strategies.

This study has contributed in terms of direction for the development of fraction sense. The findings suggest that the following strategies may help to develop pupils' fraction sense:

i)   Comparing fractions using benchmark fractions of common fractions, such as ½, ¼, zero and 1
ii)   Understanding the denominator to determine the size of equal parts
iii) Comparing fractions using unit fraction
iv)   Strategies (i) and (ii) contribute to manipulating fractions in effect of operation.

The above strategies provide guidelines for the teaching and learning of fractions, specifically for the development of fraction sense, which may enhance pupils' skills in solving mathematics problems involving fractions. This study has also revealed that the size of the values of fractions may also influence pupils' understanding of fraction sense. The pupils had difficulty in conceptualizing the value of a fraction if it was large. This result supports the findings of studies conducted by Liu, Xin and Li (2012) and Rinne, Ye and Jordan (2017). They found that children believe that when the numerator or denominator is larger, the value of the fraction also becomes larger. The same discovery was also shared by Schumacher and Malone (2017) and Wang et al. (2019) when they suggested that instruction on fractions should also focus on magnitude understanding. On the other hand, there are many strategies such as using a number line for pupils to understand the meaning of fractions and solve problems in fractions (Barbieri, Rodrigues, Dyson, & Jordan, 2019).

This study provides input for dealing with the nature of developing fraction sense. It has focused on strategies of comparing fraction values by benchmarking with some specific fractions and by referring to unit fraction. These strategies support conceptual understanding of fractions and hence contribute towards acquiring competency in fraction sense. The findings suggest that fraction sense can be developed in classroom learning. Teachers need to emphasize on strategies to deal with the numerator and denominator, especially the above-mentioned strategies, in order to develop pupils' fraction sense skills. The findings of this study also reveal some important strategies that can be applied across the constructs of fractions. The strategies are believed to be related to pupils' experiences when working on mathematics questions. Pupils who truly understand

fractions have gained a lot of experience in ratios, division, decimals and percentages. Future studies are encouraged to investigate strategies used in solving fractions and their relation to constructs such as ratio, division, decimals, percentages and others.

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